1 Basic solving with Cholesky

Solving a linear least-squares system:

$$\arg\min_x \|Ax - b\|^2$$

Set derivative equal to zero:

$$\begin{aligned} 0 &= 2A^T (Ax - b) \\ 0 &= A^T Ax - A^T b \end{aligned}$$

For comparison, with QR we do

$$\begin{aligned} 0 &= R^T Q^T QRx - R^T Qb \\ &= R^T Rx - R^T Qb \\ Rx &= Qb \\ x &= R^{-1} Qb \end{aligned}$$

But with Cholesky we do

$$\begin{aligned} 0 &= R^T RR^T Rx - R^T Rb \\ &= R^T Rx - b \\ &= Rx - R^{-T} b \\ x &= R^{-1} R^{-T} b \end{aligned}$$

2 Frontal (rank-deficient) solving with Cholesky

To do multi-frontal elimination, we decompose into rank-deficient conditionals.

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} R^T & 0 \\ S^T & C^T \end{bmatrix} \begin{bmatrix} R & S \\ 0 & C \end{bmatrix} = \begin{bmatrix} F^T F & F^T G \\ G^T F & G^T G \end{bmatrix}$$

$$R^T R = F^T F$$

$$\begin{aligned} R^T S &= F^T G \\ S &= R^{-T} F^T G \end{aligned}$$

$$
\begin{aligned}
S^T S + C^T C &= G^T G \\
G^T F R^{-1} R^{-T} F^T G + C^T C &= G^T G \\
G^T Q R R^{-1} R^{-T} R^T Q^T G + C^T C &= G^T G \\
\textbf{if } R \textbf{ is invertible, } G^T G + C^T C &= G^T G \\
C^T C &= 0
\end{aligned}
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